Diagonal Estimates for Bergman Kernels in Monomial-Type Domains

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Analysis and Applications
A Conference in Honor of Elias M. Stein
May 16, 2011

A report on joint work with Malabika Pramanik

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If $\Omega \subset \mathbb{C}^n$ is a domain, the space

$$A^2(\Omega) = \left\{ f \in L^2(\Omega) \ : \ f \text{ is holomorphic on } \Omega \right\}$$

is a closed subspace of $L^2(\Omega)$, and the Bergman projection for Ω is the orthogonal projection

$$B = B_\Omega : L^2(\Omega) \to A^2(\Omega).$$

The Berman kernel is the Schwartz kernel of this operator B, and is in fact a function $K = K_\Omega : \Omega \times \Omega \to \mathbb{C}$ so that

$$f \in L^2(\Omega) \Longrightarrow B_\Omega[f](z) = \int_\Omega f(w) K_\Omega(z, w)\, dw.$$

Let

$$\mathcal{K}(z) = \mathcal{K}_\Omega(z) := K_\Omega(z, z)$$

denote the values of the Bergman kernel on the diagonal.

Elementary properties of the Bergman kernel K_Ω:

(1) For each fixed $w \in \Omega$, the function $z \to K(z,w)$ belongs to $A^2(\Omega)$, and

$$K(z,w) = \overline{K(w,z)}.$$

(2) If $\{\varphi_n\}$ is any complete orthonormal basis of $A^2(\Omega)$, then

$$K_\Omega(z,w) = \sum_n \varphi_n(z)\overline{\varphi_n(w)}$$

with uniform convergence on compact subsets of $\Omega \times \Omega$.

(3) If $F : \Omega_1 \to \Omega_2$ is a biholomorphic mapping, then

$$K_{\Omega_1}(z,w) = JF(z)\, K_{\Omega_2}\big(F(z), F(w)\big)\, \overline{JF(w)},$$

where JF is the complex Jacobian determinant of F.

(4) If $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$, then

$$K_\Omega\big((z_1,z_2),(w_1,w_2)\big) = K_{\Omega_1}(z_1,w_1)\, K_{\Omega_2}(z_2,w_2).$$

Elementary properties of $\mathcal{K}_\Omega$, the Bergman kernel on the diagonal:

(1) The values of K on the diagonal solve an extremal problem:

$$\mathcal{K}_\Omega(z) = K_\Omega(z,z) = \sup\left\{|f(z)|^2 \,\middle|\, f \in A^2(\Omega),\ ||f|| = 1\right\}.$$

(2) If $\Omega_1 \subset \Omega_2$, and $z \in \Omega_1$ then

$$\mathcal{K}_{\Omega_2}(z) \leq \mathcal{K}_{\Omega_1}(z).$$

Examples

A. If $B_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,\big|\, \sum_{j=1}^n |z_j|^2 < 1\}$ is the unit ball, the Bergman kernel is

$$K_{B_n}(z, w) = \frac{n!}{\pi^n} \left(1 - \langle z, w \rangle\right)^{-n-1},$$

where $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$, and $\mathcal{K}_{B_n}(z) = \dfrac{n!}{\pi^n} (1 - |z|^2)^{-n-1}$.

B. If $\Delta_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,\big|\, \sup_{1 \le j \le n} |z_j| < 1\}$ is the unit polydisk, the Bergman kernel is

$$K_{\Delta_n}(z, w) = \pi^{-n} \prod_{j=1}^n (1 - z_j \overline{w_j})^{-2},$$

and $\mathcal{K}_{\Delta_n}(z) = \pi^{-n} \prod_{j=1}^n (1 - |z_j|^2)^{-2}$.

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C. Let

$$A(r,R) = \{\zeta \in \mathbb{C} \,\big|\, 0 < r < |\zeta| < R\}$$

be the annulus in $\mathbb{C}$ with inner and outer radii r and R. Suppose that $4r < 1 < \frac{R}{4}$. Then there is a constant C independent of r and R so that

$$\frac{1}{C}\left[r^2 + \frac{1}{R^2} + \frac{1}{\log\left(\frac{R}{r}\right)}\right] \leq \mathcal{K}_{A(r,R)}(1) \leq C\left[r^2 + \frac{1}{R^2} + \frac{1}{\log\left(\frac{R}{r}\right)}\right].$$

To see where these estimates come from, consider the three functions

$$f(\zeta) = r\,\zeta^{-2}, \qquad g(\zeta) \equiv \frac{1}{R}, \qquad h(\zeta) = \left[\log\left(\frac{R}{r}\right)\right]^{-\frac{1}{2}} \zeta^{-1}.$$

Then $||f||_2 \sim ||g||_2 \sim ||h||_2 \sim 1$, while

$$f(1)^2 = r^2, \qquad g(1)^2 = \frac{1}{R^2}, \qquad h(1)^2 = \frac{1}{\log\left(\frac{R}{r}\right)}.$$

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Model monomial domains

We would like to obtain uniform estimates for the Bergman kernel in domains of the form

$$\Omega = \Big\{(z, z_{n+1}) \in \mathbb{C}^{n+1} \,:\, \mathrm{Re}[z_{n+1}] > \sum_{j=1}^{d} |F_j(z_1, \dots, z_n)|^2\Big\}$$

where $\{F_1, \dots, F_d\}$ are polynomials or entire functions. However, we are only able to handle the case in which each F_j is a monomial.

Let $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_d\} \subset \mathbb{N}^n$ be a d-tuple of vectors with non-negative integer entries: $\mathbf{p}_j = (p_{j,1}, \dots, p_{j,n})$. Each $\mathbf{p}_j$ gives a monomial

$$F_{\mathbf{p}_j}(z) = z^{\mathbf{p}_j} = z_1^{p_{j,1}} \cdots z_n^{p_{j,n}}.$$

A model monomial domain is then a domain of the form

$$\Omega_{\mathcal{P}} = \Big\{(z, z_{n+1}) \in \mathbb{C}^{n+1} \,:\, \mathrm{Re}[z_{n+1}] > \sum_{j=1}^{d} |z^{\mathbf{p}_j}|^2\Big\}.$$

PROBLEM 1: Let $\Omega_{\mathcal{P}}$ be a model monomial domain. Obtain uniform estimates for $\mathcal{K}_{\Omega_{\mathcal{P}}}$, the Bergman kernel on the diagonal of $\Omega_{\mathcal{P}}$, near the boundary $\partial\Omega_{\mathcal{P}}$.

If $a \in \mathbb{C}^n$ and $\delta > 0$, the point $\left(a, \delta + \sum_{j=1}^{d} |a^{\mathbf{p}_j}|^2\right) \in \Omega_{\mathcal{P}}$. We want estimates for

$$\mathcal{K}_{\mathcal{P}}(a;\delta) := K_{\mathcal{P}}\Big((a, \delta + \sum_{j=1}^{d} |a^{\mathbf{p}_j}|^2), (a, \delta + \sum_{j=1}^{d} |a^{\mathbf{p}_j}|^2)\Big)$$

which are uniform in the base point $a \in \mathbb{C}^n$ as $\delta \searrow 0$.

Some background.

Suppose Ω is a pseudo-convex domain with smooth boundary. For $z \in \Omega$, let $\delta(z) = \inf_{w \notin \Omega} |z - w|$ be the distance to the boundary.

(a) L. Hörmander (1965) showed that if $\Omega \subset \mathbb{C}^n$ is strictly pseudo-convex, then

$$\lim_{z \to \zeta \in \partial\Omega} \delta(z)^{n+1} \mathcal{K}_\Omega(z) = \frac{n!}{4\pi^n} L(\zeta)$$

where $L(\zeta)$ is the product of the $(n-1)$ eigenvalues of the Levi form at ζ. In particular, $\mathcal{K}(z) \sim \delta(z)^{-n-1}$.

(b) C. Feffereman (1974) and L. Boutet de Monvel & J. Sjöstrand (1976) obtained a complete asymptotic expansion of $K_\Omega(z, w)$ near a boundary point $\zeta \in \Omega$ of a strictly pseudo-convex domain.

(c) D. Catlin (1989) obtained estimates for $K_\Omega(z,z)$ if $\Omega \subset \mathbb{C}^2$ is pseudo-convex of finite type.

(d) J. McNeal (1989), and N., J. Rosay, E.M. Stein, & S. Wainger (1989) obtained estimates for $K_\Omega(z,w)$ if $\Omega \subset \mathbb{C}^2$ is pseudo-convex of finite type.

(e) J. McNeal (1991) obtained estimates for $K_\Omega(z,w)$ if $\Omega \subset \mathbb{C}^n$ is a decoupled domain of finite type, and (1994) if $\Omega \subset \mathbb{C}^n$ is a convex domain of finite type.

(f) P. Charpentier & Y. Dupain (2006) obtained estimates for $K_\Omega(z,w)$ if $\Omega \subset \mathbb{C}^n$ is a locally diagonalizable domain and (2008) if Ω is geometrically separated.

In all these cases $K_\Omega(z,z) \sim \delta^{-\alpha}$ for some $\alpha > 0$.

Obtaining estimates for $\mathcal{K}_\Omega(z)$ from above:

- Let $\zeta \in \partial\Omega$, let $\vec{n}_\zeta$ be the outward unit normal to $\partial\Omega$ at ζ, and let $z = \zeta - \delta\vec{n}_\zeta$ be the point at 'height' δ above ζ. (Thus $\delta(z) = \delta$.)
- Find a polydisk $\Delta(z; r)$ with center at z with radius a small multiple of δ in the complex direction spanned by $\vec{n}_\zeta$, and poly-radii "as large as possible" in the $(n-1)$ complex directions orthogonal to $\vec{n}_\zeta$ so that $\Delta(z; r) \subset \Omega$.
- If $h \in A^2(\Omega)$, the mean value property of h gives

$$\begin{aligned} |h(z)| &\leq \frac{1}{|\Delta(z,r)|} \int_{\Delta(z,r)} |h(w)|\, dV(w) \\ &\leq \frac{1}{\sqrt{|\Delta(z,r)|}} ||h||_{L^2(\Delta(z,r))} \leq \frac{1}{\sqrt{|\Delta(z,r)|}} ||h||_{L^2(\Omega)}. \end{aligned}$$

- The extremal characterization of $\mathcal{K}(z)$ then gives

$$\mathcal{K}_\Omega(z) \leq \big|\Delta(z,r)\big|^{-1}.$$

Obtaining estimates of $\mathcal{K}_\Omega(z)$ from below:

- Construct $h \in A^2(\Omega)$ with $||h|| = 1$ which are as large as possible at z.
- In the case of model domains of the form

$$\Omega = \{(z, z_{n+1}) \in \mathbb{C}^{n+1} \ : \ \mathrm{Re}[z_{n+1}] > \sum_{j=1}^{d} |F_j(z)|^2\}$$

this can sometimes be done by considering functions of the form

$$F(z, z_{n+1}) = g(z)(\delta + z_{n+1})^{-N}$$

where g is a suitable holomorphic function of n-variables.

The Herbort Example (1983).

Let

$$P(z_1, z_2) = |z_1|^6 + |z_1 z_2|^2 + |z_2|^6$$

and let

$$\Omega_\dagger = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 \,:\, \text{Re}[z_3] > P(z_1, z_2)\Big\}.$$

Let $a = (a_1, a_2) \in \mathbb{C}^2$. If $a_1 \neq 0$ and $a_2 \neq 0$, the domain $\Omega_\dagger$ is strictly pseudo-convex at the boundary point $\big(a_1, a_2, P(a_1, a_2)\big)$. It follows from Hörmander's 1965 result that

$$\lim_{\delta \searrow 0} \delta^4 \, \mathcal{K}_{\Omega_\dagger}(a_1, a_2, \delta + P(a_1, a_2)) = c(a_1, a_2) \neq 0.$$

However if $(a_1, a_2) = (0, 0)$, Herbort (1983) showed that

$$\mathcal{K}_{\Omega_\dagger}(0, 0, \delta) \sim \delta^{-3}\Big[\log\Big(\frac{1}{\delta}\Big)\Big]^{-1}.$$

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Sketch of the proof of the bound from above:

$$\Omega_1 \times \Omega_2 = \left\{|z_1|^6 + |z_1 z_2|^2 + |z_2|^6 < \frac{1}{2}\delta\right\} \times \left\{|z_3 - \delta| < \frac{1}{2}\delta\right\} \subset \Omega_\dagger.$$

The second factor Ω_2 is a disk and contributes a factor δ^{-2}. Thus it suffices to estimate the Bergman kernel at $(0,0)$ of the domain

$$\left\{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^6 + |z_1 z_2|^2 + |z_2|^6 < \frac{1}{2}\delta\right\}.$$

If we put

$$\Omega_\dagger(\delta) := \left\{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1| < \frac{1}{6}\delta^{\frac{1}{6}},\ |z_1 z_2| < \frac{1}{6}\delta^{\frac{1}{2}},\ |z_2| < \frac{1}{6}\delta^{\frac{1}{6}}\right\},$$

then $\Omega_\dagger(\delta) \subset \Omega_1$, and it suffices to estimates the Bergman kernel of $\Omega_\dagger(\delta)$ at $(0,0)$.

If we consider polydisks centered at $(0,0)$ and contained in this domain, the maximum volume is on the order of δ, giving an estimate

$$\mathcal{K}_{\Omega_\dagger}(0,0,\delta) \lesssim \delta^{-2} \cdot \delta^{-1} = \delta^{-3}.$$

However, the domain $\Omega_\dagger(\delta)$ is invariant under rotation about each axis and it follows that if h is holomorphic on $\Omega_\dagger(\delta)$,

$$h(0,0) = |\Omega_\dagger(\delta)|^{-1} \int_{\Omega_\dagger(\delta)} h(w)\, dV(w).$$

Since

$$\big|\Omega_\dagger(\delta)\big| \sim \delta \log\left(\frac{1}{\delta}\right),$$

we can use this larger domain to obtain Herbort's better estimate

$$K_{\Omega_\dagger}(0,0,\delta) \sim \delta^{-3}\left[\log\left(\frac{1}{\delta}\right)\right]^{-1}.$$

More precise version of Problem 1:

(1a) What is the correct uniform estimate for the diagonal Bergman kernel $\mathcal{K}_{\Omega_\dagger}(z)$ which:

- (i) gives Herbort's estimate of $\delta^{-3}\left[\log(\delta^{-1})\right]^{-1}$ at height δ above the point $(0,0)$;
- (ii) gives the strictly pseudo-convex estimate δ^{-4} above nearby strictly pseudo-convex points?

(1b) Is there a geometric interpretation of such estimates?

Herbort's example suggests that instead of imbedding polydisks, one should try to imbed larger domains which are invariant under rotation (Reinhardt domains). C.H. Tiao (1999) obtained results the Bergman kernel on such domains, but needed to impose rather stringent geometric hypotheses.

In order to obtain uniform estimates, we proceed through a series of steps:

(a) reduction to the study of monomial polyhedrons in $\mathbb{C}^n$;

(b) reduction to inverse images under proper monomial mapping of monomial Reinhardt domains.

We return to the question of finding a geometric interpretation at the end.

Reduction to ‘monomial polyhedrons’.

Let

$$\Omega_{\mathcal{P}} = \Big\{\mathrm{Re}[z_{n+1}] > \sum_{j=1}^{d} |z^{\mathbf{p}_j}|^2\Big\}.$$

Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and assume for simplicity that $a_1 \cdots a_n \neq 0$.

Step 1:

There is a biholomorphic mapping with Jacobian identically 1 which carries the domain $\Omega_{\mathcal{P}}$ to

$$\Omega_{\mathcal{P},a} = \Big\{(z, z_{n+1}) \in \mathbb{C}^{n+1} \,:\, \mathrm{Re}[z_{n+1}] > \sum_{j=1}^{d} |z^{\mathbf{p}_j} - a^{\mathbf{p}_j}|^2\Big\}$$

and carries the point $\big(a, \delta + \sum_{j=1}^{d} |a^{\mathbf{p}_j}|^2\big)$ to the point (a, δ). Thus it suffices to estimate the Bergman kernel for $\Omega_{\mathcal{P},a}$ at (a, δ).

Step 2:

If

$$\widehat{\Omega}_{\mathcal{P},a} = \left\{ (z, z_{n+1}) \,:\, \sum_{j=1}^{d} |z^{\mathbf{p}_j} - a^{\mathbf{p}_j}|^2 < \frac{\delta}{2}, \quad |z_{n+1} - \delta| < \frac{\delta}{2} \right\},$$

then

$$(a, \delta) \in \widehat{\Omega}_{\mathcal{P},a} \subset \Omega_{\mathcal{P},a}.$$

Thus for upper bounds, it suffices to estimate the Bergman kernel for the domain $\widehat{\Omega}_{\mathcal{P},a}$ at (a, δ).

Step 3:

$$\widehat{\Omega}_{\mathcal{P},a} = \left\{(z, z_{n+1}) \ : \ \sum_{j=1}^{d} |z^{\mathbf{p}_j} - a^{\mathbf{p}_j}|^2 < \frac{\delta}{2}, \quad |z_{n+1} - \delta| < \frac{\delta}{2}\right\},$$

is a Cartesian product with one factor a disk. Thus it suffices to understand the Bergman kernel at the point $a \in \mathbb{C}^n$ for the domain

$$\mathcal{U}_{\mathcal{P},a}(\delta) = \left\{z \in \mathbb{C}^n \ : \ \sum_{j=1}^{d} |z^{\mathbf{p}_j} - a^{\mathbf{p}_j}|^2 < \delta\right\}.$$

Step 4:

Let

$$\mathcal{V}_{\mathcal{P},a}(\delta) = \left\{ z \in \mathbb{C}^n \,:\, \sup_{1 \le j \le d} |z^{\mathbf{p}_j} - a^{\mathbf{p}_j}|^2 < \delta \right\}.$$

Then

$$a \in \mathcal{U}_{\mathcal{P},a}(\delta) \subset \mathcal{V}_{\mathcal{P},a}(\delta) \subset \mathcal{U}_{\mathcal{P},a}(d\delta),$$

and so up to fixed multiples of δ, it suffices to understand the Bergman kernel for the domain $\mathcal{V}_{\mathcal{P},a}(\delta) \subset \mathbb{C}^n$ at the point $a \in \mathbb{C}^n$.

Step 5:

Make the biholomorphic mapping

$$F(z_1, \dots, z_n) = \left(\frac{z_1}{a_1}, \dots, \frac{z_n}{a_n}\right).$$

Then $\mathcal{V}_{\mathcal{P},a}(\delta)$ is the image under this mapping of the domain

$$\mathcal{W}_{\mathcal{P},a,\delta} := \left\{ w \in \mathbb{C}^n \,:\, |w^{\mathbf{p}_j} - 1| < \sqrt{\delta}\left(a^{\mathbf{p}_j}\right)^{-1},\ 1 \le j \le d \right\}.$$

Proposition

Let

$$\Omega_{\mathcal{P}} = \Big\{(z, z_{n+1}) \in \mathbb{C}^{n+1} \,\Big|\, Re[z_{n+1}] > P(z) := \sum_{j=1}^{d} |z^{\mathbf{p}_j}|^2\Big\}.$$

Let $\mathcal{K}_{\mathcal{P}}(z, z_{n+1})$ *be the Bergman kernel on the diagonal, and let* $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ *with each* $a_i \neq 0$. *Then*

$$\mathcal{K}_{\mathcal{P}}\big(a, \delta + P(a)\big) \leq C\,\delta^{-2}\Big[\prod_{i=1}^{n} a_i^2\Big]\mathcal{K}_{\mathcal{W}_{\mathcal{P},a,\delta}}(1)$$

where C is independent of a and δ, *and*

$$\mathcal{W}_{\mathcal{P},a,\delta} := \Big\{w \in \mathbb{C}^n \,:\, |w^{\mathbf{p}_j} - 1| < \sqrt{\delta}\big(a^{\mathbf{p}_j}\big)^{-1},\ 1 \leq j \leq d\Big\}.$$

As the base point $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ varies, the quantities

$$\delta_j = \sqrt{\delta}\big(a^{\mathbf{p}_j}\big)^{-1}, \qquad 1 \leq j \leq d,$$

vary. Thus if $\bar{\delta} = (\delta_1, \ldots, \delta_d) \in (1, \infty)^d$ and $\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_d\} \subset \mathbb{Z}^n$, consider the complex "monomial polyhedron" or "monomial ball"

$$\mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1) = \mathcal{W}_{\mathcal{P}}(\delta_1, \ldots, \delta_d; 1) = \Big\{ w \in \mathbb{C}^n \,\Big|\, |w^{\mathbf{p}_j} - 1| < \delta_j,\ 1 \leq j \leq d \Big\}.$$

More generally, for any $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, consider

$$\mathcal{W}_{\mathcal{P}}(\bar{\delta}; a) = \mathcal{W}_{\mathcal{P}}(\delta_1, \ldots, \delta_d; a) = \Big\{ w \in \mathbb{C}^n \,\Big|\, |w^{\mathbf{p}_j} - a^{\mathbf{p}_j}| < \delta_j,\ 1 \leq j \leq d \Big\}.$$

Note that if $\mathcal{W}_{\mathcal{P}}(\bar{\delta}; a)$ is not connected, $\mathcal{W}_{\mathcal{P}}(\bar{\delta}; a)$ denotes the connected component containing the point a.

Problem 2:

Estimate the Bergman kernel of the monomial polyhedron

$$\mathcal{W}_{\mathcal{P}}(\bar{\delta}; a) = \left\{ w \in \mathbb{C}^n \,\middle|\, |w^{\mathbf{p}_j} - a^{\mathbf{p}_j}| < \delta_j,\ 1 \leq j \leq d \right\}$$

at the diagonal point (a, a), with estimates that are uniform in the parameters $\bar{\delta} = \{\delta_1, \ldots, \delta_d\}$ and the point $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Note that:

- the parameters $\{\delta_1, \ldots, \delta_d\}$ can be either large or small;
- by scaling, it is easy to reduce to the case where each of the coordinates $\{a_1, \ldots, a_n\}$ is equal to zero or one.

Return to Herbort's example.

If $\mathcal{P} = \{(3,0),\ (1,1),\ (0,3)\}$ and $\bar{\delta} = (\delta_1, \delta_2, \delta_3)$, then $\mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1)$ is the domain

$$\left\{(w_1, w_2) \in \mathbb{C}^2 \,\middle|\, |w_1^3 - 1| < \delta_1,\ |w_1 w_2 - 1| < \delta_2, |w_2^3 - 1| < \delta_3\right\}.$$

We consider two special but typical cases.

Case 1: $\delta_3 \leq \delta_1 \leq \delta_2 \leq \frac{1}{100}$.

Recall that $\mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1)$ is the component containing $(1,1)$ of the set

$$\Big\{(w_1, w_2) \,:\, |w_1^3 - 1| < \delta_1,\, |w_1 w_2 - 1| < \delta_2, |w_2^3 - 1| < \delta_3\Big\}.$$

Put

$$\mathcal{R}_{\text{Big}} = \Big\{(w_1, w_2) \,\Big|\, |w_1 - 1| < 10\delta_1, \quad |w_1 w_2 - 1| < 10\delta_2\Big\},$$
$$\mathcal{R}_{\text{Small}} = \Big\{(w_1, w_2) \,\Big|\, |w_1 - 1| < \frac{1}{10}\,\delta_1, \quad |w_1 w_2 - 1| < \frac{1}{10}\,\delta_2\Big\}.$$

It is then not hard to check that

$$\mathcal{R}_{\text{Small}} \subset \mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1) \subset \mathcal{R}_{\text{Big}}.$$

Make the biholomorphic change of coordinates

$$F(w_1, w_2) = (w_1, w_1 w_2) = (u_1, u_2).$$

The Jacobian of this change of variables is uniformly bounded and bounded away from zero on $\mathcal{R}_{\text{Big}}$ since w_1 and w_2 are close to 1. The images of $\mathcal{R}_{\text{Big}}$ and $\mathcal{R}_{\text{Small}}$ are polydisks with radii comparable to δ_1 and δ_2. Thus, after a change of variables, the region $\mathcal{W}_{\mathcal{P}}(\bar{\delta}; a)$ is essentially a polydisk centered at $(1, 1)$ with radii δ_1 and δ_2. It follows that the size of the Bergman kernel for $\mathcal{W}_{\mathcal{P}}(\bar{\delta}; a)$ on the diagonal at the point $(1, 1)$ is on the order of

$$(\delta_1\, \delta_2)^{-2}.$$

Case 2: $\delta_1 \geq 1000$, $\delta_3 \geq 1000$, and $\delta_2 < \frac{1}{10}$.

Again recall that $\mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1)$ is the component containing $(1,1)$ of the set

$$\left\{(w_1, w_2) \, : \, |w_1^3 - 1| < \delta_1, \, |w_1 w_2 - 1| < \delta_2, |w_2^3 - 1| < \delta_3\right\}.$$

Note that if $(w_1, w_2) \in \mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1)$, then

$$\begin{aligned} |w_1| &= \frac{|w_1 w_2|}{|w_2|} = \frac{|1 + (w_1 w_2 - 1)|}{|w_2|} \geq \frac{1 - |w_1 w_2 - 1|}{|w_2|} \\ &\geq \frac{1}{2|w_2|} = \frac{1}{2|w_2^3 - 1 + 1|^{\frac{1}{3}}} \geq \frac{1}{4}\delta_3^{-\frac{1}{3}}. \end{aligned}$$

This time, put

$$\mathcal{R}_{\text{Big}} = \left\{(w_1, w_2) \, : \, \frac{1}{10}\delta_3^{-\frac{1}{3}} < |w_1| < 10\delta_1^{\frac{1}{3}}, \quad |w_1 w_2 - 1| < 10\delta_2, \right\},$$

$$\mathcal{R}_{\text{Small}} = \left\{(w_1, w_2) \,\middle|\, 10\delta_3^{-\frac{1}{3}} < |w_1| < \frac{1}{10}\delta_1^{\frac{1}{3}}, \quad |w_1 w_2 - 1| < \frac{1}{10}\delta_2, \right\}.$$

It is then not hard to check that $\mathcal{R}_{\text{Small}} \subset \mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1) \subset \mathcal{R}_{\text{Big}}$.

Again make the biholomorphic change of coordinates

$$F(w_1, w_2) = (w_1, w_1 w_2) = (u_1, u_2).$$

The Jacobian of this change of variables is w_1, which equals 1 at the point $(1, 1)$.

This time the images of $\mathcal{R}_{\text{Big}}$ and $\mathcal{R}_{\text{Small}}$ are the Cartesian product of a disk centered at 1 of radius comparable to δ_2, with an annulus whose outer radius is comparable to $\delta_1^{\frac{1}{3}}$ and whose inner radius is comparable to $\delta_3^{-\frac{1}{3}}$.

It follows that in this case the size of the Bergman kernel for $\mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1)$ on the diagonal at the point $(1, 1)$ is again on the order of

$$\delta_2^{-2} \Big[\log(\delta_1 \delta_3) \Big]^{-1}.$$

Explicit estimates in Herbort-type examples.

Consider the following generalization of Herbort's example:

$$\Omega_{\dagger} = \Big\{(z_1, z_2, z_3) \,:\, \mathrm{Re}[z_3] > |z_1|^{2m} + |z_1 z_2|^2 + |z_2|^{2n}\Big\},$$

and consider the point

$$z = \big(a, b, \delta + |a|^{2m} + |ab|^2 + |b|^{2n} + it\big)$$

which is at height δ above the boundary.

Theorem

If $\mathcal{K}_\Omega$ is the Bergman kernel for $\Omega_\dagger$ on the diagonal, if $(a,b) \in \mathbb{C}^2$ with $|a|^2 + |b|^2 \leq 1$, and if $z = \big(a, b, \delta + |a|^{2m} + |ab|^2 + |b|^{2n} + it\big)$, then

$$\mathcal{K}_{\Omega_\dagger}(z) \approx \begin{cases} \dfrac{|a|^{2m}}{\delta^4} & \text{if} \quad \delta^{\frac{1}{2m}} \lesssim |a| \text{ and } |b| \leq |a|, \\[2ex] \dfrac{|b|^{2n}}{\delta^4} & \text{if} \quad \delta^{\frac{1}{2n}} \lesssim |b| \text{ and } |a| \leq |b|, \\[2ex] \dfrac{1}{\delta^3}\left[\dfrac{|a|^2}{\delta^{\frac{1}{m}}} + \dfrac{|b|^2}{\delta^{\frac{1}{n}}} + \dfrac{1}{\log^+\left(\frac{\delta^{\frac{1}{2m}} + \delta^{\frac{1}{2n}}}{|ab|}\right)}\right] & \text{if} \quad \begin{cases} |a| \lesssim \delta^{\frac{1}{2m}}, \\ |b| \lesssim \delta^{\frac{1}{2n}}, \\ \delta^{\frac{1}{2}} \lesssim |ab|, \end{cases} \\[4ex] \dfrac{1}{\delta^3}\left[\dfrac{|a|^2}{\delta^{\frac{1}{m}}} + \dfrac{|b|^2}{\delta^{\frac{1}{n}}} + \dfrac{1}{\log^+\left(\frac{1}{\delta}\right)}\right] & \text{if} \quad \begin{cases} |a| \lesssim \delta^{\frac{1}{2m}}, \\ |b| \lesssim \delta^{\frac{1}{2n}}, \\ |ab| \lesssim \delta^{\frac{1}{2}}. \end{cases} \end{cases}$$

Return to the general case.

A domain $\mathcal{U} \subset \mathbb{C}^n$ is a Reinhardt domain if

$$(z_1, \ldots, z_n) \in \mathcal{U} \quad \Longrightarrow \quad (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \mathcal{U}.$$

A domain $\mathcal{R}_{\mathcal{Q}}(\bar{\eta})$ is a rational, monomial-type Reinhardt domain if

$$\mathcal{Q} = \{\mathbf{q}_1, \ldots, \mathbf{q}_s\} \subset \mathbb{Q}^n, \quad \mathbf{q}_k = (q_{k,1}, \ldots, q_{k,n});$$
$$\bar{\eta} = (\eta_1, \ldots, \eta_s) \in (0, \infty)^s;$$

and

$$\mathcal{R}_{\mathcal{Q}}(\bar{\eta}) = \mathcal{R}_{\mathcal{Q}}(\eta_1, \ldots, \eta_s) = \Big\{ z \in \mathbb{C}^n \ : \ \prod_{j=1}^{n} |z_j|^{q_{k,j}} < \eta_k, \ 1 \leq k \leq s \Big\}.$$

Let $0 < \epsilon_1 < 1 < \epsilon_2$. A domain $\Omega \subset \mathbb{C}^n$ is (ϵ_1, ϵ_2)-approximated by the monomial-type Reinhardt domain $\mathcal{R}_{\mathcal{Q}}(\bar{\eta})$ if

$$\mathcal{R}_{\mathcal{Q}}(\epsilon_1 \eta_1, \ldots, \epsilon_1 \eta_s) \subset \Omega \subset \mathcal{R}_{\mathcal{Q}}(\epsilon_2 \eta_1, \ldots, \epsilon_2 \eta_s).$$

Theorem

Let $\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_d\} \subset \mathbb{N}^d$ *be a spanning set of vectors, and put*

$$\mathcal{W}_{\mathcal{P}}(\bar{\delta}; 1) = \left\{ w \in \mathbb{C}^n \,\middle|\, |w^{\mathbf{p}_j} - 1| < \delta_j,\; 1 \leq j \leq d \right\}$$

be a complex monomial polyhedron. There there exists

(a) *a monomial mapping* $\Phi = (m_1, \ldots, m_n) : \mathbb{C}^n \to \mathbb{C}^n$ *(i.e. each component function* m_j *is a monomial in* $z_1, \ldots, z_n$*);*

(b) *a monomial-type Reinhardt domain* $\mathcal{R}_{\mathcal{Q}}(\bar{\eta}) \subset \mathbb{C}^n$

(c) *absolute constants* $0 < \epsilon_1 < 1 < \epsilon_2$*,*

so that $\Phi\big(\mathcal{W}_{\mathcal{P}}(\bar{\delta}, 1)\big)$ *is* (ϵ_1, ϵ_2)*-approximated by* $\mathcal{R}_{\mathcal{Q}}(\bar{\eta})$*; i.e.*

$$\mathcal{R}_{\mathcal{Q}}(\epsilon_1 \bar{\eta}) \subset \Phi\big(\mathcal{W}_{\mathcal{P}}(\bar{\delta}, 1)\big) \subset \mathcal{R}_{\mathcal{Q}}(\epsilon_2 \bar{\eta}).$$

The point of this result is the following:

- Let $z \in \mathcal{R}_{\mathcal{Q}}(\epsilon_1\bar{\eta}) \subset \mathcal{R}_{\mathcal{Q}}(\epsilon_2\bar{\eta})$. We show that there is a constant $C(\epsilon_1, \epsilon_2)$ so that

$$\mathcal{K}_{\mathcal{R}_{\mathcal{Q}}(\epsilon_2\bar{\eta})}(z) \leq \mathcal{K}_{\mathcal{R}_{\mathcal{Q}}(\epsilon_1\bar{\eta})}(z) \leq C(\epsilon_1, \epsilon_2)\,\mathcal{K}_{\mathcal{R}_{\mathcal{Q}}(\epsilon_2\bar{\eta})}(z).$$

- It follows that

$$\mathcal{K}_{\mathcal{R}_{\mathcal{Q}}(\epsilon_2\bar{\eta})}(z) \leq \mathcal{K}_{\Phi\left(\mathcal{W}_{\mathcal{P}}(\bar{\delta},1)\right)}(z) \leq C(\epsilon_1, \epsilon_2)\,\mathcal{K}_{\mathcal{R}_{\mathcal{Q}}(\epsilon_2\bar{\eta})}(z).$$

- Thus we obtain diagonal estimates for the Bergman kernel for $\Phi\big(\mathcal{W}_{\mathcal{P}}(\bar{\delta}, 1)\big)$ in terms of diagonal estimates for rational, monomial-type Reinhardt domains.

This leads to two further problems:

(Problem 2) Understand the relationship between the Bergman kernel of a domain $\mathcal{W}$ and the Bergman kernel of its image $\Phi(\mathcal{W})$ where Φ is a proper holomorphic monomial mapping.

(Problem 3) Obtain estimates for $\mathcal{K}_\Omega$ when Ω is a rational, monomial-type Reinhardt domain.

The solution of Problem 2 involves an orthogonal decomposition $A^2(\mathcal{W}) = \bigoplus A^2_\chi(\mathcal{W})$ into closed subspaces parameterized by characters χ of a finite abelian group G. Each Hilbert space $A^2_\chi(\mathcal{W})$ is then isometric with the space $A^2(\Phi(\mathcal{W}), \omega_\chi(w)dV(w))$ where ω_χ is a weight function on $\Phi(\mathcal{W})$. This allows us to estimate the Bergman kernel for $\mathcal{W}$ in terms of a sum of weighted Bergman kernels for $\Phi(\mathcal{W})$.

In the remainder of the talk, we focus on Problem 3.

Let $\mathcal{R} \subset \mathbb{C}^n$ be a Reinhardt domain, and put

$$|\mathcal{R}| = \big\{(|z_1|, \ldots, |z_n|) \,:\, (z_1, \ldots, z_n) \in \mathcal{R}\big\};$$
$$|\mathcal{R}^*| = \big\{(x_1, \ldots, x_n) \in |\mathcal{R}| \,:\, x_j \neq 0,\ 1 \leq j \leq n\big\}$$
$$\Sigma_{\mathcal{R}} = \big\{(t_1, \ldots, t_n) \in \mathbb{R}^n \,:\, (e^{t_1}, \ldots, e^{t_n}) \in |\mathcal{R}^*|\big\} = \log\big(|\mathcal{R}^*|\big).$$

For each $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, let $F_{\mathbf{m}}(z) = z^{\mathbf{m}} = z_1^{m_1} \cdots z_n^{m_n}$.

Lemma

- *$\mathcal{R}$ is pseudo-convex if and only if $\Sigma_{\mathcal{R}}$ is convex.*
- *If h is holomorphic on $\mathcal{R}$, then $h(z) = \sum_{\mathbf{m} \in \mathbb{Z}^n} c_{\mathbf{m}}\, z^{\mathbf{m}}$, with absolute and uniform convergence on compact subsets of $\mathcal{R}$.*
- *The collection of functions $\{F_{\mathbf{m}}(z) = z^{\mathbf{m}} : F_{\mathbf{m}} \in A^2(\mathcal{R})\}$ is a complete orthogonal basis for $A^2(\mathcal{R})$.*
- *The Bergman kernel for $\mathcal{R}$ is given by*

$$K_{\mathcal{R}}(z, w) = \sum_{\mathbf{m} \in \mathbb{Z}^n} \frac{F_{\mathbf{m}}(z)\overline{F_{\mathbf{m}}(w)}}{||F_{\mathbf{m}}||_{L^2}^2} = \sum_{\mathbf{m} \in \mathbb{Z}^n} \frac{z^{\mathbf{m}}\overline{w}^{\mathbf{m}}}{||F_{\mathbf{m}}||_{L^2}^2}.$$

In the last formula for the Bergman kernel $K_{\mathcal{R}}(z, w)$, we actually sum only over those $\mathbf{m}$ for which $||F_{\mathbf{m}}||_{L^2} < \infty$. This set can be characterized as follows. For each $0 \neq \mathbf{y} \in \mathbb{R}^n$, let

$$M(\mathbf{y}) = \sup_{\mathbf{t}\in\Sigma_{\mathcal{R}}} \langle \mathbf{t}, \mathbf{y} \rangle,$$
$$\Gamma(\Sigma) = \left\{\mathbf{y} \in \mathbb{R}^n \,:\, M(\mathbf{y}) < +\infty\right\}.$$

Proposition

(a) *The set* $\Gamma(\Sigma)$ *is a convex cone in* $\mathbb{R}^n$*.*

(b) $||F_{\mathbf{m}}||_{L^2(\mathcal{R})} < \infty$ *if and only if* $\mathbf{m}$ *belongs to the interior of* $\Gamma(\Sigma)$*.*

Thus

$$K_{\mathcal{R}}(z, w) = \sum_{\mathbf{m}\in\mathbb{Z}^n\cap\mathrm{int}(\Gamma(\Sigma))} \frac{F_{\mathbf{m}}(z)\overline{F_{\mathbf{m}}(w)}}{||F_{\mathbf{m}}||^2_{L^2}} = \sum_{\mathbf{m}\in\mathbb{Z}^n\cap\mathrm{int}(\Gamma(\Sigma))} \frac{z^{\mathbf{m}}\overline{w}^{\mathbf{m}}}{||F_{\mathbf{m}}||^2_{L^2}}.$$

We introduce the following notation:
If $0 \neq \mathbf{y} \in \Gamma(\Sigma)$, let

$$\Pi_{\mathbf{y}} = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle = \sup_{\mathbf{s} \in \Sigma} \langle \mathbf{s}, \mathbf{y} \rangle\}$$

be the supporting hyperplane to Σ which is perpendicular to $\mathbf{y}$.

For $\mathbf{t} \in \Sigma$, let $\rho_{\mathbf{y}}(\mathbf{t})$ be the perpendicular distance from $\mathbf{t}$ to $\Pi_{\mathbf{y}}$. Thus

$$\rho_{\mathbf{y}}(\mathbf{t}) = |\mathbf{y}|^{-1} \sup_{\mathbf{s} \in \Sigma} \langle \mathbf{s} - \mathbf{t}, \mathbf{y} \rangle.$$

For $\mathbf{m} \in \Gamma(\Sigma)$, let $V(\mathbf{m})$ denote the volume of the 'cap' $C(\mathbf{m})$ of Σ of thickness $|\mathbf{m}|^{-1}$ in the direction of $\mathbf{m}$:

$$C(\mathbf{m}) = \left\{\mathbf{t} \in \Sigma \,:\, \rho_{\mathbf{m}}(\mathbf{t}) \leq |m|^{-1}\right\} = \left\{\mathbf{t} \in \Sigma \,:\, \sup_{\mathbf{s} \in \Sigma} \langle \mathbf{s} - \mathbf{t}, \mathbf{m} \rangle \leq 1\right\};$$

$$V(\mathbf{m}) = \left|\left\{\mathbf{t} \in \Sigma \,:\, \rho_{\mathbf{m}}(\mathbf{t}) \leq |m|^{-1}\right\}\right| = \left|\left\{\mathbf{t} \in \Sigma \,:\, \sup_{\mathbf{s} \in \Sigma} \langle \mathbf{s} - \mathbf{t}, \mathbf{m} \rangle \leq 1\right\}\right|.$$

Proposition

Let $\mathcal{R}$ be a log-convex Reinhardt domain, and let $\Sigma = \log(|\mathcal{R}^*|)$. Let

$$z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \mathcal{R}$$

so that $\mathbf{x} = (x_1, \ldots, x_n) \in \Sigma$. Then

$$\begin{aligned} \mathcal{K}_{\mathcal{R}}(z) &= e^{-2\langle \mathbf{1},\mathbf{x}\rangle} \sum_{\mathbf{m}\in\mathbb{Z}^n\cap int(\Gamma(\Sigma))} e^{-2|\mathbf{m}|\rho_{\mathbf{m}}(\mathbf{x})} \left[\int_{\Sigma} e^{-2|\mathbf{m}|\rho_{\mathbf{m}}(s)}\, ds\right]^{-1} \\ &\approx e^{-2\langle \mathbf{1},\mathbf{x}\rangle} \sum_{\mathbf{m}\in\mathbb{Z}^n\cap int(\Gamma(\Sigma))} e^{-2|\mathbf{m}|\rho_{\mathbf{m}}(\mathbf{x})} V(\mathbf{m})^{-1} \qquad (1) \end{aligned}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and the constants implied by the symbol "$\approx$" are independent of $\mathcal{R}$ and z

Note that if $\mathbf{m} \in \mathbb{Z}^n \cap \mathrm{int}(\Gamma(\Sigma))$, then

$$|\mathbf{m}|\rho_{\mathbf{m}}(\mathbf{x}) \leq 1 \quad \iff \quad \sup_{\mathbf{s}\in\Sigma}\langle \mathbf{s} - \mathbf{x}, \mathbf{m}\rangle \leq 1,$$

which means that $\mathbf{x}$ belongs to the cap $C(\mathbf{m})$.

Recall that if $\mathbf{T} \subset \mathbb{R}^n$, the polar set of $\mathbf{T}$ is

$$\mathbf{T}^* = \{\mathbf{y} \in \mathbb{R}^n \,:\, \sup_{\mathbf{t}\in\mathbf{T}}\langle \mathbf{t}, \mathbf{y}\rangle \leq 1\}.$$

Thus if $\mathbf{x} \in \Sigma$, then

$$|\mathbf{m}|\rho_{\mathbf{m}}(\mathbf{x}) \leq 1 \quad \iff \quad \mathbf{m} \in (\Sigma_{\mathbf{x}})^*$$

where

$$\Sigma_{\mathbf{x}} = \Sigma - \{\mathbf{x}\} = \{\mathbf{t} - \mathbf{x} \,:\, \mathbf{t} \in \Sigma\}.$$

We show that the main contribution to the series in (1) comes from the set of $\mathbf{m} \in \mathbb{Z}^n \cap \mathrm{int}(\Gamma(\Sigma))$ for which $|\mathbf{m}|\rho_{\mathbf{m}}(\mathbf{x}) \leq 1$.

Theorem

Let $\mathcal{R}$ be a rational monomial-type Reinhardt domain, and let

$$z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \mathcal{R}.$$

Let $\Sigma = \log(|\mathcal{R}|)$*, so that* $\mathbf{x} = (x_1, \ldots, x_n) \in \Sigma$*, and let*

$$V(\mathbf{x}) = \inf\left\{|V(\mathbf{m})| \ : \mathbf{m} \in \mathbb{Z}^n \cap int(\Gamma(\Sigma)), \ \mathbf{x} \in C(\mathbf{m})\right\}.$$

Then

$$\mathcal{K}_{\mathcal{R}}(z) \approx e^{-2\langle \mathbf{1},\mathbf{x}\rangle} \sum_{\mathbf{m}:\rho_{\mathbf{m}}(\mathbf{x}) \leq |\mathbf{m}|^{-1}} V(\mathbf{m})^{-1}$$

$$\approx e^{-2\langle \mathbf{1},\mathbf{x}\rangle} \left(1 + \left|(\Sigma_{\mathbf{x}})^*\right|\right) \left[V(\mathbf{x})\right]^{-1}.$$